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ON THE PROBLEM OF FLEXURE OF ANISOTROPIC CYLINDRICAL SHELLS†

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Abstract—The problem of determining stresses and deformations in anisotropic, elastic, axially homogeneous thin-walled shells subjected to equal and opposite transverse end forces, and additional moments to assure overall equilibrium, is studied. The solution procedure is on the basis of a decomposition of stresses and strains in terms of a portion dependent linearly on the axial coordinate and a portion independent of the axial coordinate. The most significant aspect of the work has to do with the analysis of the effect of anisotropy of the material. The general formulae of the theory are illustrated for a class of shells consisting of an "ordinary" material. Here explicit formulae are obtained for certain types of open– as well as closed-cross-section shells.

INTRODUCTION

THIS paper considers elastic axially homogeneous anisotropic cylindrical shells subject to transverse end forces, together with the corresponding bending moments to assure overall equilibrium, as a problem on extension of the classical St. Venant theory of flexure. The differential equations of equilibrium and compatibility are special cases of general linear shell equations given by Günther [1], and the constitutive equations are special cases of equations given by Reissner [2]. Important characteristics of these equations are the inclusion of moments turning about the normals to the middle surface of the shell, and incorporation of the effect of transverse shear deformation. The effect of anisotropy on the distribution of stresses, strains and displacements is the particular concern of this paper.

In attempting a solution of the flexure problem, we start with a formulation of the overall equilibrium conditions for the shell acted upon by transverse end forces and the associated equilibrium end bending moments. We conclude from these that in its simplest form the solution of the problem should consist of two portions. The first portion, due to the flexural bending moments, depends linearly on the axial coordinate. The second portion due to the flexural forces is independent of the axial coordinate. Investigation of the general system of differential equations and its associated boundary conditions reveals that in fact the two indicated portions are all that is necessary for the solution of the problem of axially homogeneous cylindrical shell flexure, upon appeal to an appropriate version of St. Venant's principle.

In the following we first indicate the procedure of decomposing the problem into the two mentioned portions. We then proceed to indicate the solution procedure for both portions. The result for the axially dependent portion is directly given upon making use

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of an appropriate interpretation for the result of the problem of pure bending, stretching and twisting, as recently presented by Reissner and Tsai [3]. The result for the axially independent portion is then obtained as function of the quantities of the axially dependent portion.

FUNDAMENTAL EQUATIONS

We take as curvilinear coordinates the circumferential arc length s and the axial distance z. The coefficients of the linear arc element are $\alpha_s = \alpha_z = 1$, while the curvatures are $1/R_{ss} = 1/R = \mathrm{d}\varphi/\mathrm{d}s = k$ and $1/R_{zz} = 1/R_{sz} = 0$. Designating stress resultants by N_{ss}, N_{sz}, N_{zs}, N_{zz}, Q_s, Q_z and stress couples by M_{ss}, M_{sz}, M_{zs}, M_{zz}, P_s, P_z, with corresponding strain resultants ε, γ and strain couples $\varkappa$, λ, we have as six equilibrium equations and six compatibility equations in accordance with Günther [1],

$$N_{ss,s}+N_{zs,z}+kQ_s = 0, \qquad \varkappa_{zz,s}-\varkappa_{sz,z}-k\lambda_z = 0, \tag{1a, b}$$

$$Q_{s,s}+Q_{z,z}-kN_{ss} = 0, \qquad \lambda_{z,s}-\lambda_{s,z}+k\varkappa_{zz} = 0, \tag{2a, b}$$

$$M_{ss,s}+M_{zs,z}-Q_s = 0, \qquad \varepsilon_{zz,s}-\varepsilon_{sz,z}-\lambda_z = 0, \tag{3a, b}$$

$$N_{sz,s}+N_{zz,z} = 0, \qquad \varkappa_{zs,s}-\varkappa_{ss,z} = 0, \tag{4a, b}$$

$$M_{sz,s}+M_{zz,z}-kP_s-Q_z = 0, \qquad \varepsilon_{zs,s}-\varepsilon_{ss,z}+k\gamma_z-\lambda_s = 0, \tag{5a, b}$$

$$P_{s,s}+P_{z,z}+kM_{sz}+N_{sz}-N_{zs} = 0, \qquad \gamma_{z,s}-\gamma_{s,z}-k\varepsilon_{zs}+\varkappa_{zs}-\varkappa_{sz} = 0. \tag{6a, b}$$

The twelve equations (1a)–(6b) which involve twelve stress quantities and twelve strain quantities are here complemented by twelve constitutive equations, written in the form

$$\{N_{ss}N_{zz}N_{sz}N_{zs}Q_sQ_zM_{ss}M_{zz}M_{sz}M_{zs}P_sP_z\} = A\{\varepsilon_{ss}\varepsilon_{zz}\varepsilon_{sz}\varepsilon_{zs}\gamma_s\gamma_z\varkappa_{ss}\varkappa_{zz}\varkappa_{sz}\varkappa_{zs}\lambda_s\lambda_z\}, \tag{7}$$

where A is a 12×12 matrix, the elements of which are given functions of the arc length coordinates.

In addition we have twelve strain–displacement relations involving translational displacements u_s, u_z, w and rotational displacements φ_s, φ_z, ω. These relations are

$$\begin{aligned} &\varepsilon_{ss} = u_{s,s}+kw, \qquad \varepsilon_{zz} = u_{z,z}, \qquad \gamma_s = w_{,s}+\varphi_s-ku_s, \\ &\varkappa_{ss} = \varphi_{s,s}, \qquad \varkappa_{zz} = \varphi_{z,z}, \qquad \lambda_z = \omega_{,z}, \end{aligned} \tag{8a}$$

and

$$\begin{aligned} &\varepsilon_{sz} = u_{z,s}-\omega, \qquad \varepsilon_{zs} = u_{s,z}+\omega, \qquad \gamma_z = w_{,z}+\varphi_z, \\ &\varkappa_{sz} = \varphi_{z,s}-k\omega, \qquad \varkappa_{zs} = \varphi_{s,z}, \qquad \lambda_s = \omega_{,s}+k\varphi_z. \end{aligned} \tag{8b}$$

Having stated the appropriate system of differential equations, we now consider the associated boundary conditions for ends z = constant of the shell and for boundaries s = constant of the shell.

There are altogether twelve boundary conditions for s = constant. For clarity's sake we consider these conditions separately for open- and closed-cross-section tubes, as follows. For the open-cross-section tube with longitudinal boundaries $s = s_1$ and $s = s_2$, we assume that the longitudinal boundaries are traction-free. This means that we prescribe the vanishing of the three stress resultants N_{ss}, N_{sz}, Q_s, and of the three stress couples M_{ss}, M_{sz}, P_s for $s = s_1$ and for $s = s_2$.

For the closed-cross-section tube, we assume that the stress measures N_{ss}, N_{sz}, Q_s, M_{ss}, M_{sz}, P_s and the displacement measures u_s, u_z, w, φ_s, φ_z, ω altogether are twelve univalued quantities. Since our interest is essentially in the determination of stress and strain quantities, we replace the six displacement univaluedness conditions by six conditions expressed in terms of strain quantities. We find with the help of the strain displacement relations (8a, b) that the univaluedness of displacement conditions are equivalent to two conditions of univalued ε_{zs}, γ_z and to four strain integral relations

$$0 = \oint \varkappa_{ss}\,\mathrm{d}s, \tag{9a}$$

$$0 = \oint (\varkappa_{ss}y + \varepsilon_{ss}x' + \gamma_s y')\,\mathrm{d}s, \tag{9b}$$

$$0 = \oint (\varkappa_{ss}x - \varepsilon_{ss}y' + \gamma_s x')\,\mathrm{d}s, \tag{9c}$$

$$0 = \oint [\varepsilon_{sz} - (\varkappa_{sz}y' + \lambda_s x')x + (\varkappa_{sz}x' - \lambda_s y')y]\,\mathrm{d}s. \tag{9d}$$

We now turn to the consideration of boundary conditions for sections z = constant. We appeal to St. Venant's principle and replace the local conditions of prescribed edge stresses by overall equilibrium conditions as follows,

$$0 = \int N_{zz}\,\mathrm{d}s \tag{10a}$$

$$Q_y z = \int (N_{zz}y - M_{zz}x' + P_z y')\,\mathrm{d}s, \tag{10b}$$

$$Q_x z = \int (N_{xx}x + M_{zz}y' + P_z x')\,\mathrm{d}s, \tag{10c}$$

$$0 = \int [M_{zs} + (N_{zs}y' - Q_z x')x - (N_{zs}x' + Q_z y')y]\,\mathrm{d}s, \tag{10d}$$

$$Q_y = \int (N_{zs}y' - Q_z x')\,\mathrm{d}s, \tag{10e}$$

$$Q_x = \int (N_{zs}x' + Q_z y')\,\mathrm{d}s. \tag{10f}$$

In this, Q_x and Q_y are the prescribed overall transverse forces, $Q_x z$ and $Q_y z$ are the prescribed overall bending moments, and equations (10a, d) are the statement of vanishing of axial force and of axial torque.

We note that the six integral relations (10a) to (10f) are equivalent to four only, consisting of the relations (10a, b, c, d). The two relations (10e, f) may be shown, by means of an introduction of (4a) to (6a) into the z-derivatives of (10b, c) and by suitable integrations by part, upon making use of the previously indicated boundary conditions for N_{ss}, Q_s, N_{sz}, M_{ss}, M_{sz}, P_s for s = constant, to be equivalent to equations (10b, c).

DECOMPOSITION OF STRESSES AND STRAINS

Given the form of the boundary conditions (10) and the fact that the differential equations (1) to (6) together with (7) and the remaining boundary conditions are homogeneous, we realize the possibility of the following decomposition of the state of stress and strain

$$(N, Q, M, P, \varepsilon, \gamma, \varkappa, \lambda)(s, z) = z\,.\,(N^*, Q^*, M^*, P^*, \varepsilon^*, \gamma^*, \varkappa^*, \lambda^*)(s) + (\bar{N}, \bar{Q}, \bar{M}, \bar{P}, \bar{\varepsilon}, \bar{\gamma}, \bar{\varkappa}, \bar{\lambda})(s). \tag{11}$$

Introduction of (11) into (1) to (6) and (10a, b, c, d) leaves as system of equations for the z-dependent quantities

$$N^*_{ss,s} + kQ^*_s = 0, \qquad \varkappa^*_{zz,s} - k\lambda^*_z = 0, \qquad (1a^*, b^*)$$

$$Q^*_{s,s} - kN^*_{ss} = 0, \qquad \lambda^*_{z,s} + k\varkappa^*_{zz} = 0, \qquad (2a^*, b^*)$$

$$M^*_{ss,s} - Q^*_s = 0, \qquad \varepsilon^*_{zz,s} - \lambda^*_z = 0, \qquad (3a^*, b^*)$$

$$N^*_{sz,s} = 0, \qquad \varkappa^*_{zs,s} = 0, \qquad (4a^*, b^*)$$

$$M^*_{sz,s} - kP^*_s - Q^*_z = 0, \qquad \varepsilon^*_{zs,s} + k\gamma^*_z - \lambda^*_s = 0, \qquad (5a^*, b^*)$$

$$P^*_{s,s} + kM^*_{sz} + N^*_{sz} - N^*_{zs} = 0, \qquad \gamma^*_{z,s} - k\varepsilon^*_{zs} + \varkappa^*_{zs} - \varkappa^*_{sz} = 0, \qquad (6a^*, b^*)$$

and

$$0 = \int N^*_{zz}\,\mathrm{d}s, \qquad (10a^*)$$

$$Q_y = \int (N^*_{zz}y - M^*_{zz}x' + P^*_z y')\,\mathrm{d}s, \qquad (10b^*)$$

$$Q_x = \int (N^*_{zz}x + M^*_{zz}y' + P^*_z x')\,\mathrm{d}s, \qquad (10c^*)$$

$$0 = \int [M^*_{zs} + (N^*_{zs}y' - Q^*_z x')x - (N^*_{zs}x' + Q^*_z y')y]\,\mathrm{d}s. \qquad (10d^*)$$

At the same time we obtain as system of equations for the z-independent quantities

$$\bar{N}_{ss,s} + k\bar{Q}_s = -N^*_{zs}, \qquad \bar{\varkappa}_{zz,s} - k\bar{\lambda}_z = \varkappa^*_{sz}, \qquad (1\bar{a}, \bar{b})$$

$$\bar{Q}_{s,s} - k\bar{N}_{ss} = -Q^*_z, \qquad \bar{\lambda}_{z,s} + k\bar{\varkappa}_{zz} = \lambda^*_s, \qquad (2\bar{a}, \bar{b})$$

$$\bar{M}_{ss,s} - \bar{Q}_s = -M^*_{zs}, \qquad \bar{\varepsilon}_{zz,s} - \bar{\lambda}_z = \varepsilon^*_{sz}, \qquad (3\bar{a}, \bar{b})$$

$$\bar{N}_{sz,s} = -N^*_{zz}, \qquad \bar{\varkappa}_{zs,s} = \varkappa^*_{ss}, \qquad (4\bar{a}, \bar{b})$$

$$\bar{M}_{sz,s} - k\bar{P}_s - \bar{Q}_z = -M^*_{zz}, \qquad \bar{\varepsilon}_{zs,s} + k\bar{\gamma}_z - \bar{\lambda}_s = \varepsilon^*_{ss}, \qquad (5\bar{a}, \bar{b})$$

$$\bar{P}_{s,s} + k\bar{M}_{sz} + \bar{N}_{sz} - \bar{N}_{zs} = P^*_z, \qquad \bar{\gamma}_{z,s} - k\bar{\varepsilon}_{zs} + \bar{\varkappa}_{zs} - \bar{\varkappa}_{sz} = \gamma^*_s, \qquad (6\bar{a}, \bar{b})$$

and

$$0 = \int \bar{N}_{zz}\, ds, \tag{10ā}$$

$$0 = \int (\bar{N}_{zz} y - \bar{M}_{zz} x' + \bar{P}_z y')\, ds, \tag{10b̄}$$

$$0 = \int (\bar{N}_{zz} x + \bar{M}_{zz} y' + \bar{P}_z x')\, ds, \tag{10c̄}$$

$$0 = \int [\bar{M}_{zs} + (\bar{N}_{zs} y' - \bar{Q}_z x')x - (\bar{N}_{zs} x' + \bar{Q}_z y')y]\, ds. \tag{10d̄}$$

It is obvious that equations (1a*) to (6b*) and (10a*, b*, c*, d*) are of exactly the same form as equations (1a) to (6b) and (16a, b, c, d) in [3] except that the places of the stretching force N, the bending moments M_x and M_y and of the twisting moment T in equations (16a, b, c, d) in [3] are taken over by 0, Q_y, Q_x and 0, respectively. In addition to this, the constitutive equations and the boundary conditions for the starred quantities are also equivalent to the corresponding formulas in [3]. From this follows that the results for the problem of determining the *starred* measures may be stipulated to be directly given in the same form as the corresponding results for the problem of pure bending, stretching and twisting in [3], upon replacing N, M_x, M_y, T by 0, Q_y, Q_x, 0, respectively.

We next turn to the determination of the *barred* quantities on the basis of equations (1ā) to (6b̄) and (10ā, b̄, c̄, d̄), together with the appropriate boundary conditions and constitutive equations.

FIRST INTEGRALS OF EQUATIONS FOR BARRED QUANTITIES

Let s_0 be a specified arc length coordinate. We begin by integrating equations (4ā, b̄) in the form

$$\bar{N}_{sz} = \bar{S} - \int_{s_0}^{s} N^*_{zz}\, ds, \qquad \bar{\varkappa}_{zs} = \bar{\alpha} + \int_{s_0}^{s} \varkappa^*_{ss}\, ds. \tag{12a, b}$$

where $\bar{S}$ and $\bar{\alpha}$ are constants of integration.

Next, we integrate equations (1ā, b̄) and (2ā, b̄), with the help of the relation $(\ )_{,s} = k(\ )_{,\varphi}$, with equations (5a*) to (6b*), and with appropriate integrations by part. In this way, we find the important result that we can express $\bar{N}_{ss}$, $\bar{\varkappa}_{zz}$, $\bar{Q}_s$ and $\bar{\lambda}_z$ explicitly as

$$\bar{N}_{ss} = \bar{N}_x x' - \bar{N}_y y' - S^* r_t - P^*_s, \qquad \bar{\varkappa}_{zz} = \bar{\varkappa}_y y' - \bar{\varkappa}_x x' + \alpha^* r_t + \gamma^*_z, \tag{13a, b}$$

$$\bar{Q}_s = \bar{N}_x y' + \bar{N}_y x' - S^* r_n - M^*_{sz}, \qquad \bar{\lambda}_z = \bar{\varkappa}_x y' + \bar{\varkappa}_y x' - \alpha^* r_n + \varepsilon^*_{zs}, \tag{14a, b}$$

In this $r_n \equiv xy' - x'y$, $r_t \equiv xx' + yy'$; $S^* \equiv N^*_{sz}$ and $\alpha^* \equiv \varkappa^*_{zs}$ are given constants; and $\bar{N}_x$, $\bar{N}_y$, $\bar{\varkappa}_x$ and $\bar{\varkappa}_y$ are four additional constants of integration.

We now introduce (14a, b) into ($3\bar{a}$, $\bar{b}$), and obtain with $\bar{M}$ and $\bar{\varepsilon}$ as two further constants of integration,

$$\bar{M}_{ss} = \bar{N}_x y + \bar{N}_y x + \bar{M} - \int_{s0}^{s} (M_{sz}^* + M_{zs}^* + S^* r_n)\,\mathrm{d}s, \tag{15a}$$

$$\bar{\varepsilon}_{zz} = \bar{\varkappa}_x y + \bar{\varkappa}_y x + \bar{\varepsilon} + \int_{s0}^{s} (\varepsilon_{sz}^* + \varepsilon_{zs}^* - \alpha^* r_n)\,\mathrm{d}s. \tag{15b}$$

Having the system of first integrals (12a) to (15b), we are left with four first-order differential equations ($5\bar{a}$) to ($6\bar{b}$), which involve the eight quantities $\bar{M}_{sz}$, $\bar{P}_s$, $\bar{Q}_z$, $\bar{N}_{zs}$, $\bar{\varepsilon}_{zs}$, $\bar{\gamma}_z$, $\bar{\lambda}_s$ and $\bar{\varkappa}_{sz}$, and which can not in general be integrated explicitly.

BOUNDARY CONDITIONS FOR BARRED QUANTITIES

Equations ($1\bar{a}$) to ($6\bar{b}$) are a twelfth-order system of *ordinary* differential equations. We look for twelve boundary conditions. We have, in accordance with the previously given fundamental equations, four stress integral relations ($10\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$) and twelve boundary conditions for s = constant. Among these, four of the boundary conditions for s = constant are extraneous, leaving us twelve required boundary conditions. We show this by considering separately the cases of open- and closed-cross-section tubes.

For the case of closed-cross-section tubes, we have that S^*, M_{sz}^* and P_s^* are univalued, and $\oint N_{zz}^*\,\mathrm{d}s$ and $\oint (M_{sz}^* + M_{zs}^* + S^* r_n)\,\mathrm{d}s$ vanish. Introducing this into (12a) to (15a) we find that $\bar{N}_{sz}$, $\bar{N}_{ss}$, $\bar{Q}_s$ and $\bar{M}_{ss}$ are the measures which automatically satisfy the univaluedness conditions for arbitrary s. Upon excluding the conditions for these four quantities we are left with twelve boundary conditions which consist of the four conditions of univaluedness of $\bar{M}_{sz}$, $\bar{P}_s$, $\bar{\varepsilon}_{zs}$ and $\bar{\gamma}_z$, the four stress integral relations ($10\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$), and the four barred strain integral relations in (9a, b, c, d), written here in the form

$$\oint \bar{\varkappa}_{ss}\,\mathrm{d}s = 0, \tag{9$\bar{a}$}$$

$$\oint (\bar{\varkappa}_{ss} y + \bar{\varepsilon}_{ss} x' + \bar{\gamma}_s y')\,\mathrm{d}s = 0, \tag{9$\bar{b}$}$$

$$\oint (\bar{\varkappa}_{ss} x - \bar{\varepsilon}_{ss} y' + \bar{\gamma}_s x')\,\mathrm{d}s = 0, \tag{9$\bar{c}$}$$

$$\oint [\bar{\varepsilon}_{sz} - (\bar{\varkappa}_{sz} y' + \bar{\lambda}_s x')x + (\bar{\varkappa}_{sz} x' - \bar{\lambda}_s y')y]\,\mathrm{d}s = 0. \tag{9$\bar{d}$}$$

We note for subsequent reference the following important transformation of equations ($10\bar{d}$) and ($9\bar{d}$)

$$\int (\bar{M}_{sz} + \bar{M}_{zs} + \bar{S} r_n)\,\mathrm{d}s = \int \left(r_n \int_{s0}^{s} N_{zz}^*\,\mathrm{d}s + r_t M_{zz}^* - r_n P_z^* \right) \mathrm{d}s, \tag{10$\bar{d}'$}$$

and

$$\oint(\bar{\varepsilon}_{zs}+\bar{\varepsilon}_{sz}-\bar{\alpha}r_n)\,\mathrm{d}s = \oint\left(r_n\int_{s_0}^{s}\varkappa_{ss}^*\,\mathrm{d}s-r_t\varepsilon_{ss}^*-r_n\gamma_s^*\right)\mathrm{d}s. \tag{9$\bar{\mathrm{d}}'$}$$

Further simplifications are possible upon noting the geometrical formula $2A_e = \oint r_n\,\mathrm{d}s$.

It remains to formulate the boundary conditions for open-cross-section tubes. We let $s_0 = s_1$ and use the facts that $S^* = 0$ throughout, and $M_{sz}^* = P_s^* = 0$ for $s = s_1$. The vanishing of $\bar{N}_{sz}$, $\bar{N}_{ss}$, $\bar{Q}_s$ and $\bar{M}_{ss}$ for $s = s_1$ in (12a) to (15a) is then equivalent to

$$\bar{S} = \bar{N}_x = \bar{N}_y = \bar{M} = 0, \tag{16}$$

and we are left with the explicit results

$$\begin{aligned} \bar{N}_{sz} &= -\int_{s_1}^{s} N_{zz}^*\,\mathrm{d}s, & \bar{N}_{ss} &= -P_s^*, \\ \bar{Q}_s &= -M_{sz}^*, & \bar{M}_{ss} &= -\int_{s_1}^{s}(M_{sz}^*+M_{zs}^*)\,\mathrm{d}s. \end{aligned} \tag{17}$$

It is apparent that the boundary conditions of vanishing $\bar{N}_{sz}$, $\bar{N}_{ss}$, $\bar{Q}_s$, $\bar{M}_{ss}$ for $s = s_2$ are satisfied automatically because of the boundary conditions for the starred quantities for open tubes involving

$$\int_{s_1}^{s_2} N_{zz}^*\,\mathrm{d}s = \int_{s_1}^{s_2}(M_{sz}^*+M_{zs}^*)\,\mathrm{d}s = 0 \quad \text{and} \quad M_{sz}^* = P_s^* = 0 \quad \text{for} \quad s = s_2.$$

Recognizing this, we are again left with twelve boundary conditions involving the four results $\bar{S} = \bar{N}_x = \bar{N}_y = \bar{M} = 0$ in (16), four conditions $\bar{M}_{sz} = \bar{P}_s = 0$ for $s = s_1$ and $s = s_2$, and the four conditions (10$\bar{\mathrm{a}}$, $\bar{\mathrm{b}}$, $\bar{\mathrm{c}}$) and (10$\bar{\mathrm{d}}'$).

We note in conjunction with the result (17) that the quantities $\bar{N}_{sz}$, $\bar{N}_{ss}$, $\bar{Q}_s$ and $\bar{M}_{ss}$ together with the result that N_{sz}^*, N_{ss}^*, Q_s^* and M_{ss}^* vanish throughout, and with the decomposition (11) provide explicit results for the actual quantities N_{sz}, N_{ss}, Q_s and M_{ss} for open tubes, namely

$$\begin{aligned} N_{sz} &= -\int_{s_1}^{s} N_{zz}^*\,\mathrm{d}s, & N_{ss} &= -P_s^*, \\ Q_s &= -M_{sz}^*, & M_{ss} &= -\int_{s_1}^{s}(M_{sz}^*+M_{zs}^*)\,\mathrm{d}s. \end{aligned} \tag{18}$$

Equations (18) show that N_{sz}, N_{ss}, Q_s and M_{ss} are always independent of z. Furthermore, the relation $Q_s+M_{sz}^* = 0$ in (18) means an automatic satisfaction of the Kirchhoff boundary condition $Q_s+M_{sz,z} = 0$ for the case that the transverse shearing strain γ_z is stipulated to vanish.

CALCULATION OF DISPLACEMENTS

In what follows we consider displacements of the points of the cylindrical shell surface for the problem of flexure. As for the problems of pure stretching, bending and twisting we are led to distinguish two types of displacements, those which translate and rotate the cross section of the shell as a rigid body and those which represent a distortion of the cross section, axially and laterally. In both types, two different classes of elements are involved. The class of "ordinary" elements which also occur for orthotropic shells and the class of "unusual" elements which are due to the effect of non-orthotropy.

We start to calculate displacements by considering the strain displacement relations (8a, b) which with the help of the decomposition (11) assume the form

$$\begin{aligned} &\varepsilon^*_{ss}z+\bar{\varepsilon}_{ss} = u_{s,s}+kw, \qquad \varepsilon^*_{zz}z+\bar{\varepsilon}_{zz} = u_{z,z}, \qquad \gamma^*_s z+\bar{\gamma}_s = \varphi_s+w_{,s}-ku_s,\\ &\varkappa^*_{ss}z+\bar{\varkappa}_{ss} = \varphi_{s,s}, \qquad \varkappa^*_{zz}z+\bar{\varkappa}_{zz} = \varphi_{z,z}, \qquad \lambda^*_z z+\bar{\lambda}_z = \omega_{,z}, \end{aligned} \tag{19a}$$

and

$$\begin{aligned} &\varepsilon^*_{sz}z+\bar{\varepsilon}_{sz} = u_{z,s}-\omega \qquad \varepsilon^*_{zs}z+\bar{\varepsilon}_{zs} = u_{s,z}+\omega, \qquad \gamma^*_z z+\bar{\gamma}_z = \varphi_z+w_{,z},\\ &\varkappa^*_{sz}z+\bar{\varkappa}_{sz} = \varphi_{z,s}-k\omega, \qquad \varkappa^*_{zs}z+\bar{\varkappa}_{zs} = \varphi_{s,z}, \qquad \lambda^*_s z+\bar{\lambda}_s = \omega_{,s}+k\varphi_z. \end{aligned} \tag{19b}$$

In these $\bar{\varkappa}_{zs}$, $\bar{\varkappa}_{zz}$, $\bar{\lambda}_z$, $\bar{\varepsilon}_{zz}$ are explicitly given by (12b) to (15b) and $\varkappa^*_{zs}$, $\varkappa^*_{zz}$, λ^*_z, ε^*_{zz}, with α^*, $\varkappa^*_x$, $\varkappa^*_y$, ε^* as four constants, are

$$\begin{aligned} &\varkappa^*_{zs} = \alpha^*, \qquad \varkappa^*_{zz} = \varkappa^*_y y'-\varkappa^*_x x',\\ &\lambda^*_z = \varkappa^*_x y'+\varkappa^*_y x', \qquad \varepsilon^*_{zz} = \varkappa^*_x y+\varkappa^*_y x+\varepsilon^*. \end{aligned} \tag{20}$$

We now introduce (20) together with the first strain integrals (12b) to (15b) and with the strain differential equations (5b*, $\bar{5}$) and (6b*, $\bar{5}$) into the strain displacement relations (19a, b). An observation of the z-independence of all starred and barred strain components leads to the following expressions for the six components of rotational and translational displacement

$$\varphi_s = \tfrac{1}{2}\alpha^* z^2+\bar{\alpha}z+\int \varkappa_{ss}\,ds, \tag{21a}$$

$$\begin{aligned} \varphi_z = {} & \tfrac{1}{2}(\varkappa^*_y y'-\varkappa^*_x x')z^2+(\bar{\varkappa}_y y'-\bar{\varkappa}_x x'+\alpha^* r_t)z+\gamma_z+\bar{\alpha}[(x-x_0)x'+(y-y_0)y']\\ & +r_t\int \varkappa^*_{ss}\,ds-x'\int(\varkappa^*_{ss}x-\varepsilon^*_{ss}y'+\gamma^*_s x')\,ds-y'\int(\varkappa^*_{ss}y+\varepsilon^*_{ss}x'+\gamma^*_s y')\,ds, \end{aligned} \tag{21b}$$

$$\begin{aligned} \omega = {} & \tfrac{1}{2}(\varkappa^*_x y'+\varkappa^*_y x')z^2+(\bar{\varkappa}_x y'+\bar{\varkappa}_y x'-\alpha^* r_n)z+\varepsilon_{zs}-\bar{\alpha}[(x-x_0)y'-(y-y_0)x']\\ & -r_n\int \varkappa^*_{ss}\,ds+y'\int(\varkappa^*_{ss}x-\varepsilon^*_{ss}y'+\gamma^*_s x')\,ds-x'\int(\varkappa^*_{ss}y+\varepsilon^*_{ss}x'+\gamma^*_s y')\,ds, \end{aligned} \tag{21c}$$

and, with $(\varkappa_{ss}, \varepsilon_{ss}, \gamma_s) \equiv (\varkappa^*_{ss}, \varepsilon^*_{ss}, \gamma^*_s)z+(\bar{\varkappa}_{ss}, \bar{\varepsilon}_{ss}, \bar{\gamma}_s)$,

$$\begin{aligned} u_z = {} & \tfrac{1}{2}(\varkappa^*_x y+\varkappa^*_y x+\varepsilon^*)z^2+(\bar{\varkappa}_x y+\bar{\varkappa}_y x+\bar{\varepsilon})z+\bar{\alpha}(x_0 y-y_0 x)+\int(\varepsilon_{sz}+\varepsilon_{zs}-\varkappa_{zs}r_n)\,ds\\ & +\int(r_t\varepsilon^*_{ss}+r_n\gamma^*_s)\,ds+y\int(\varkappa^*_{ss}x-\varepsilon^*_{ss}y'+\gamma^*_s x')\,ds-x\int(\varkappa^*_{ss}y+\varepsilon^*_{ss}x'+\gamma^*_s y')\,ds, \end{aligned} \tag{22a}$$

$$\begin{aligned} u_s = {} & -\tfrac{1}{6}(\varkappa^*_x y'+\varkappa^*_y x')z^3-\tfrac{1}{2}(\bar{\varkappa}_x y'+\bar{\varkappa}_y x'-\alpha^* r_n)z^2+\bar{\alpha}[(x-x_0)y'-(y-y_0)x']z\\ & +r_n\int \varkappa_{ss}\,ds-y'\int(\varkappa_{ss}x-\varepsilon_{ss}y'+\gamma_s x')\,ds+x'\int(\varkappa_{ss}y+\varepsilon_{ss}x'+\gamma_s y')\,ds, \end{aligned} \tag{22b}$$

$$w = -\tfrac{1}{6}(\varkappa_y^* y' - \varkappa_x^* x')z^3 - \tfrac{1}{2}(\bar{\varkappa}_y y' - \bar{\varkappa}_x x' + \alpha^* r_t)z^2 - \bar{\alpha}[(x-x_0)x' + (y-y_0)y']z$$

$$- r_t \int \varkappa_{ss}\,\mathrm{d}s + x' \int (\varkappa_{ss}x - \varepsilon_{ss}y' + \gamma_s x')\,\mathrm{d}s + y' \int (\varkappa_{ss}y + \varepsilon_{ss}x' + \gamma_s y')\,\mathrm{d}s. \qquad (22c)$$

We note that the translational components u_s and w can be resolved into two displacement components v_x and v_y in x and y directions upon introducing

$$v_x = u_s x' + w y', \qquad v_y = u_s y' - w x'. \qquad (23)$$

The result is given by

$$v_x = -\tfrac{1}{6}\varkappa_y^* z^3 - \tfrac{1}{2}(\bar{\varkappa}_y + \alpha^* y)z^2 - \bar{\alpha}(y-y_0)z - y\int \varkappa_{ss}\,\mathrm{d}s + \int (\varkappa_{ss}y + \varepsilon_{ss}x' + \gamma_s y')\,\mathrm{d}s, \qquad (24a)$$

$$v_y = -\tfrac{1}{6}\varkappa_x^* z^3 - \tfrac{1}{2}(\bar{\varkappa}_x - \alpha^* x)z^2 + \bar{\alpha}(x-x_0)z + x\int \varkappa_{ss}\,\mathrm{d}s - \int (\varkappa_{ss}x - \varepsilon_{ss}y' + \gamma_s x')\,\mathrm{d}s. \qquad (24b)$$

In equations (24a, b) the terms without integral sign represent displacements with the tube cross section translating and rotating as a rigid body and the terms with integral signs represent the effect of distortion. Furthermore, since $(\varkappa_{ss}, \varepsilon_{ss}, \gamma_s) \equiv (\varkappa_{ss}^*, \varepsilon_{ss}^*, \gamma_s^*)z + (\bar{\varkappa}_{ss}, \bar{\varepsilon}_{ss}, \bar{\gamma}_s)$, the axially linear dependence of the effect of distortion involves the well-known effect of transverse shear on the deflection of the beam, as stated by Timoshenko and Goodier [4] for the problem of isotropic sheet flexure. The remaining measures, involving $\bar{\varkappa}_{ss}$, $\bar{\varepsilon}_{ss}$ and $\bar{\gamma}_s$, are the unusual terms which vanish for the case of orthotropic materials. We also find from the terms without integral sign in (24a, b) that rigid body translation and rotation involve terms proportional to z^3, z^2 and z. In these, the z^3-terms represent the deflection due to flexural bending, and the z-terms represent the deflection due to axial twisting produced by the flexural forces† [6]. We designate the z^2-terms as "unusual" since they represent deflections which vanish for the case of orthotropic materials.

We note in conjunction with the above observation for v_x and v_y that the axial displacement u_z also involves both ordinary and unusual terms. The z^2 and z-independent terms are of course the ordinary terms representing the displacement due to bending and warping. The term linear in z is the unusual term which is again due to anisotropy because the coefficient of this term is, with equation (15b), the axially homogeneous strain measure $\bar{\varepsilon}_z$ which comes from the constitutive coupling coefficients for anisotropic tubes.

METHOD OF SOLUTION FOR THE GENERAL CASE

Equations (12a) to (15b) express the four stress quantities $\bar{N}_{sz}$, $\bar{N}_{ss}$, $\bar{Q}_s$, $\bar{M}_{ss}$ and the four strain quantities $\bar{\varkappa}_{zs}$, $\bar{\varkappa}_{zz}$, $\bar{\lambda}_z$, $\bar{\varepsilon}_{zz}$ in terms of eight constants of integration. Equations ($5\bar{a}$) to ($6\bar{b}$) are four first-order differential equations allowing us to introduce four additional constants of integration. The total of twelve constants of integration is to be determined in terms of integrals involving starred quantities by means of the four stress integral relations ($10\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}'$) together with the four univaluedness conditions for $\bar{M}_{sz}$, $\bar{P}_s$, $\bar{\varepsilon}_{zs}$, $\bar{\gamma}_z$ and the four strain integral relations ($9\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}'$) for closed-cross-section tubes, or to-

† For the case of orthotropic tubes, axial twisting may be eliminated by letting the flexural forces act through the *center of twist*, Reissner and Tsai [5].

gether with the eight traction conditions along the longitudinal edges, $\bar{N}_{sz} = \bar{N}_{ss} = \bar{Q}_s = \bar{M}_{ss} = 0$ for $s = s_1$ and $\bar{M}_{sz} = \bar{P}_s = 0$ for $s = s_1$, and for $s = s_2$, for open-cross-section tubes. In what follows we wish to indicate the solution procedure for obtaining the barred quantities for the case of the closed-cross-section shell, in such a way that the open-cross-section shell solution is included as a special case.

Considering the solution procedure for the general case of anisotropy, we first rewrite the constitutive equations (7) for the barred contributions in the form

$$\begin{bmatrix} \bar{N}_{zz} \\ \bar{\varepsilon}_{ss} \\ \bar{\gamma}_s \\ \bar{M}_{zz} \\ \bar{\varkappa}_{ss} \\ \bar{P}_z \\ \bar{\varepsilon}_{sz} \\ \bar{M}_{zs} \end{bmatrix} = B_{88} \begin{bmatrix} \bar{\varepsilon}_{zz} \\ \bar{N}_{ss} \\ \bar{Q}_s \\ \bar{\varkappa}_{zz} \\ \bar{M}_{ss} \\ \bar{\lambda}_z \\ \bar{N}_{sz} \\ \bar{\varkappa}_{zs} \end{bmatrix} + B_{84} \begin{bmatrix} \bar{\varepsilon}_{zs} \\ \bar{\gamma}_z \\ \bar{M}_{sz} \\ \bar{P}_s \end{bmatrix}, \quad \begin{bmatrix} \bar{N}_{zs} \\ \bar{Q}_z \\ \bar{\varkappa}_{sz} \\ \bar{\lambda}_s \end{bmatrix} = B_{48} \begin{bmatrix} \bar{\varepsilon}_{zz} \\ \bar{N}_{ss} \\ \bar{Q}_s \\ \bar{\varkappa}_{zz} \\ \bar{M}_{ss} \\ \bar{\lambda}_z \\ \bar{N}_{sz} \\ \bar{\varkappa}_{zs} \end{bmatrix} + B_{44} \begin{bmatrix} \bar{\varepsilon}_{zs} \\ \bar{\gamma}_z \\ \bar{M}_{sz} \\ \bar{P}_s \end{bmatrix}. \qquad (25a, b)$$

In this, the eight quantities $\bar{\varepsilon}_{zz}$, $\bar{\varkappa}_{zz}$, $\bar{\lambda}_z$, $\bar{\varkappa}_{zs}$ and $\bar{M}_{ss}$, $\bar{N}_{ss}$, $\bar{Q}_s$, $\bar{N}_{sz}$ are given by the first integrals (12) to (15).

We further write the differential equations (5ā, b̄) and (6ā, b̄) in the form

$$\begin{bmatrix} \bar{N}_{zs} \\ \bar{Q}_z \\ \bar{\varkappa}_{sz} \\ \bar{\lambda}_s \end{bmatrix} = \begin{bmatrix} \bar{P}'_s \\ \bar{M}'_{sz} \\ \bar{\gamma}'_z \\ \bar{\varepsilon}'_{zs} \end{bmatrix} + k \begin{bmatrix} \bar{M}_{sz} \\ -\bar{P}_s \\ -\bar{\varepsilon}_{zs} \\ \bar{\gamma}_z \end{bmatrix} + \begin{bmatrix} \bar{N}_{sz} \\ 0 \\ \bar{\varkappa}_{zs} \\ 0 \end{bmatrix} + \begin{bmatrix} P_z^* \\ M_{zz}^* \\ -\gamma_s^* \\ -\varepsilon_{ss}^* \end{bmatrix}, \qquad (26a, b)$$

and use this to transform (25b) into

$$\begin{bmatrix} \bar{P}'_s \\ \bar{M}'_{sz} \\ \bar{\gamma}'_z \\ \bar{\varepsilon}'_{zs} \end{bmatrix} = B_{44}^* \begin{bmatrix} \bar{M}_{sz} \\ \bar{P}_s \\ \bar{\varepsilon}_{zs} \\ \bar{\gamma}_z \end{bmatrix} + B_{48}^* \begin{bmatrix} \bar{\varepsilon}_{zz} \\ \bar{N}_{ss} \\ \bar{Q}_s \\ \bar{\varkappa}_{zz} \\ \bar{M}_{ss} \\ \bar{\lambda}_z \\ \bar{N}_{sz} \\ \bar{\varkappa}_{zs} \end{bmatrix} - \begin{bmatrix} P_z^* \\ M_{zz}^* \\ -\gamma_s^* \\ -\varepsilon_{ss}^* \end{bmatrix}. \qquad (27)$$

We now have (27) as a fourth-order system of simultaneous differential equations for $\bar{P}_s$, $\bar{M}_{sz}$, $\bar{\gamma}_z$ and $\bar{\varepsilon}_{zs}$. The nonhomogeneous part of this involves the *known* starred quantities P_z^*, M_{zz}^*, γ_s^*, ε_{ss}^* and the *given* barred quantities $\bar{\varepsilon}_{zz}$, $\bar{\varkappa}_{zz}$, $\bar{\lambda}_z$, $\bar{\varkappa}_{zs}$, $\bar{M}_{ss}$, $\bar{N}_{ss}$, $\bar{Q}_s$, $\bar{N}_{sz}$. Equations (27) are to be solved subject to the four boundary conditions of univaluedness of $\bar{P}_s$, $\bar{M}_{sz}$, $\bar{\gamma}_z$ and $\bar{\varepsilon}_{zs}$ for the case of the closed-cross-section shell.

Having solved (27) we must introduce the result into the constitutive equations (25a, b) to express, with the help of the first integrals (12a) to (15b), all quantities on the left-hand sides as functions of the eight constants of integration $\bar{S}$, $\bar{N}_x$, $\bar{N}_y$, $\bar{M}$, $\bar{\alpha}$, $\bar{\varkappa}_x$, $\bar{\varkappa}_y$, $\bar{\varepsilon}$, as well as of certain functions of the starred quantities appearing explicitly and implicitly in (27). These expressions in turn are to be introduced into the set (10ā, b̄, c̄, d̄′) together with (9ā, b̄, c̄, d̄′), thereby producing a system of eight simultaneous linear equations for the determination of the eight constants $\bar{S}$, $\bar{N}_x$, $\bar{N}_y$, $\bar{M}$, $\bar{\alpha}$, $\bar{\varkappa}_x$, $\bar{\varkappa}_y$, $\bar{\varepsilon}$ in terms of integrals of starred quantities.

The fourth-order system (27) is reduced to a zeroth-order system if we stipulate, in conformity with observations in [3] that the shearing strain component $\bar{\gamma}_z$ and the moment stress resultant $\bar{P}_s$ may be set equal to zero. With this, equation (27) reduces to two simultaneous equations for $\bar{M}_{sz}$ and $\bar{\varepsilon}_{zs}$ of the form

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = B_{22}^{**} \begin{bmatrix} \bar{M}_{sz} \\ \bar{\varepsilon}_{zs} \end{bmatrix} + B_{28}^{**} \begin{bmatrix} \bar{\varepsilon}_{zz} \\ \bar{N}_{ss} \\ \bar{Q}_s \\ \bar{\varkappa}_{zz} \\ \bar{M}_{ss} \\ \bar{\lambda}_z \\ \bar{N}_{sz} \\ \bar{\varkappa}_{zs} \end{bmatrix} - \begin{bmatrix} P_z^* \\ -\gamma_s^* \end{bmatrix}. \tag{28a}$$

At the same time, the quantities $\bar{Q}_z$ and $\bar{\lambda}_s$ are given by

$$\bar{Q}_z = \bar{M}'_{sz} + M_{zz}^*, \qquad \bar{\lambda}_s = \bar{\varepsilon}'_{zs} - \varepsilon_{ss}^*. \tag{28b}$$

In what follows we extend this general consideration to describe the solution procedure for the case for which the constitutive equations are taken in the form of the so-called conventional case.

METHOD OF SOLUTION FOR CONVENTIONAL CASE

In using the words "conventional cases" we have in mind cases for which the constitutive equations are of the form

$$\{N_{zz}N_T N_{ss} M_{zz} M_T M_{ss}\} = A_c\{\varepsilon_{zz}\varepsilon_T\varepsilon_{ss}\varkappa_{zz}\varkappa_T\varkappa_{ss}\}, \tag{29}$$

together with

$$\gamma_z = \gamma_s = 0, \qquad P_s = P_z = 0, \tag{30a, b}$$

$$\varepsilon_{zs} = \varepsilon_{sz} \equiv \varepsilon_T, \qquad M_{sz} = M_{zs} \equiv M_T, \tag{31a, b}$$

and with the definitions

$$2N_T = N_{sz}+N_{zs}, \qquad 2\varkappa_T = \varkappa_{zs}+\varkappa_{sz}, \tag{32a, b}$$

Having equations (30a, b) we have from the equilibrium equation ($6\bar{a}$), $\bar{N}_{zs} = \bar{N}_{sz} + k\bar{M}_T$, and from the compatibility equation ($6\bar{b}$), $\bar{\varkappa}_{sz} = \bar{\varkappa}_{zs} - k\varepsilon_T$, and therewith

$$\bar{N}_T = \bar{N}_{sz}+\tfrac{1}{2}k\bar{M}_T, \qquad \bar{\varkappa}_T = \bar{\varkappa}_{zs}-\tfrac{1}{2}k\bar{\varepsilon}_T. \tag{33a, b}$$

We now use the simplified subscript notations $(N_a, M_a, \varepsilon_a, \varkappa_a) \equiv (N_{aa}, M_{aa}, \varepsilon_{aa}, \varkappa_{aa})$. With this we introduce (33) into the barred part of (29) and this with the help of a suitable transformation gives

$$\begin{bmatrix} \bar{N}_z \\ \bar{M}_z \\ \bar{M}_T \end{bmatrix} = B_1 \begin{bmatrix} \bar{\varepsilon}_z \\ \bar{\varkappa}_z \\ 2\bar{\varkappa}_{zs} \end{bmatrix} + B_2 \begin{bmatrix} \bar{M}_s \\ -\bar{N}_s \\ \bar{N}_{sz} \end{bmatrix}, \qquad \begin{bmatrix} \bar{\varkappa}_s \\ -\bar{\varepsilon}_s \\ 2\bar{\varepsilon}_T \end{bmatrix} = B_3 \begin{bmatrix} \bar{\varepsilon}_z \\ \bar{\varkappa}_z \\ 2\bar{\varkappa}_{zs} \end{bmatrix} + B_4 \begin{bmatrix} \bar{M}_s \\ -\bar{N}_s \\ \bar{N}_{sz} \end{bmatrix}, \tag{34a, b}$$

where $\bar{\varepsilon}_z$, $\bar{\varkappa}_z$, $\bar{\varkappa}_{zs}$, $\bar{M}_s$, $\bar{N}_s$ and $\bar{N}_{sz}$ are as in equations ($12\bar{a}$) to ($15\bar{b}$).

In order to simplify the result for the barred quantities, we rewrite equations ($12\bar{a}$) to ($15\bar{b}$) in the matrix form

$$\begin{bmatrix} \bar{\varepsilon}_z \\ \bar{\varkappa}_z \\ 2\bar{\varkappa}_{zs} \end{bmatrix} = Q_2 \begin{bmatrix} \bar{\alpha} \\ \bar{\varepsilon} \\ \bar{\varkappa}_x \\ \varkappa_y \end{bmatrix} + f_2, \qquad \begin{bmatrix} \bar{M}_s \\ -\bar{N}_s \\ \bar{N}_{sz} \end{bmatrix} = Q_1 \begin{bmatrix} \bar{S} \\ \bar{M} \\ \bar{N}_x \\ \bar{N}_y \end{bmatrix} - f_1. \tag{35a, b}$$

where

$$Q_i = \begin{bmatrix} 0 & 1 & y & x \\ 0 & 0 & -x' & y' \\ i & 0 & 0 & 0 \end{bmatrix}, \qquad (i = 1, 2), \tag{36}$$

and where

$$f_1 = \begin{bmatrix} \int_{s0}^{s} (2M_T^* + S^* r_n)\,\mathrm{d}s \\ -S^* r_t \\ \int_{s0}^{s} N_z^*\,\mathrm{d}s \end{bmatrix}, \qquad f_2 = \begin{bmatrix} \int_{s0}^{s} (2\varepsilon_T^* - \alpha^* r_n)\,\mathrm{d}s \\ \alpha^* r_t \\ 2\int_{s0}^{s} \varkappa_s^*\,\mathrm{d}s \end{bmatrix}. \tag{37a, b}$$

Having equations (34) and (35), it remains to determine the eight constants of integrations $\bar{\varepsilon}$, $\bar{\varkappa}_x$, $\bar{\varkappa}_y$, $\bar{\alpha}$, $\bar{M}$, $\bar{N}_x$, $\bar{N}_y$ and $\bar{S}$.

Open-cross-section shells

For an open-cross-section shell, the boundary conditions of vanishing traction for $s = s_1$ and $s = s_2$ give $\bar{M} = \bar{N}_x = \bar{N}_y = \bar{S} = 0$. The corresponding result for $\bar{N}_{sz}$, $\bar{N}_s$ and $\bar{M}_s$ is given in (17) and can, with $M^*_{sz} = M^*_{zs} \equiv M^*_T$ and $P^*_s = 0$, and with $S^* = 0$, be written as

$$\{\bar{M}_s - \bar{N}_s \bar{N}_{sz}\} = -f_1 = -\int_{s_1}^{s} \{2M^*_T \, 0 \, N^*_z\} \, \mathrm{d}s. \tag{38}$$

It now remains only to determine the four constants of integration $\bar{\varepsilon}$, $\bar{\varkappa}_x$, $\bar{\varkappa}_y$ and $\bar{\alpha}$ in $\bar{\varepsilon}_z$, $\bar{\varkappa}_z$ and $\bar{\varkappa}_{zs}$. Evidently, this will be accomplished upon introducing (34a), (35a) and (38) into equations (10ā, b̄, c̄, d̄′), written here in the matrix form

$$\int_{s_1}^{s_2} Q_2^T \begin{bmatrix} \bar{N}_z \\ \bar{M}_z \\ \bar{M}_T \end{bmatrix} \mathrm{d}s = \int_{s_1}^{s_2} g_1 \, \mathrm{d}s, \tag{39}$$

where Q_2^T is transpose of Q_2 and g_1 is a vector given by

$$g_1 = \left\{ r_n \int_{so}^{s} N^*_z \, \mathrm{d}s + r_t M^*_z \;\; 0 \;\; 0 \;\; 0 \right\}. \tag{40a}$$

Equations (39) directly give a system of equations for $\bar{\alpha}$, $\bar{\varepsilon}$, $\bar{\varkappa}_x$ and $\bar{\varkappa}_y$ in terms of integrals of starred quantities of the form

$$\int_{s_1}^{s_2} Q_2^T B_1 Q_2 \, \mathrm{d}s \begin{bmatrix} \bar{\alpha} \\ \bar{\varepsilon} \\ \bar{\varkappa}_x \\ \bar{\varkappa}_y \end{bmatrix} = \int_{s_1}^{s_2} [g_1 - Q_2^T(B_1 f_2 - B_2 f_1)] \, \mathrm{d}s. \tag{41}$$

Closed-cross-section shell

We now have the overall equilibrium equations (10ā, b̄, c̄, d̄′), with $\bar{P}_z = 0$ and $\bar{M}_{zs} = \bar{M}_{sz} \equiv \bar{M}_T$, written in the form

$$\oint Q_2^T \begin{bmatrix} \bar{N}_z \\ \bar{M}_z \\ \bar{M}_T \end{bmatrix} \mathrm{d}s + \begin{bmatrix} 2A_e \bar{S} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \oint g_1 \, \mathrm{d}s, \tag{42}$$

and this, with (34a) and (35a, b), and with four-by-four matrices C_i, becomes

$$C_1 \begin{bmatrix} \bar{\alpha} \\ \bar{\varepsilon} \\ \bar{\varkappa}_x \\ \bar{\varkappa}_y \end{bmatrix} + C_2 \begin{bmatrix} \bar{S} \\ \overline{M} \\ \overline{N}_x \\ \overline{N}_y \end{bmatrix} = \oint [g_1 - Q_2^T (B_1 f_2 - B_2 f_1)] \, \mathrm{d}s. \tag{43}$$

At the same time, we define

$$g_2 = \left\{ r_n \int_{s_0}^{s} \varkappa_s^* \, \mathrm{d}s - r_t \varepsilon_s^* \;\; 0 \;\; 0 \;\; 0 \right\}, \tag{40b}$$

to write the strain integral relations ($9\bar{\mathrm{a}}$, $\bar{\mathrm{b}}$, $\bar{\mathrm{c}}$, $\bar{\mathrm{d}}'$), with $\bar{\gamma}_s = 0$ and $\bar{\varepsilon}_{sz} = \bar{\varepsilon}_{zs} \equiv \bar{\varepsilon}_T$, into the form

$$\oint Q_1^T \begin{bmatrix} \bar{\varkappa}_s \\ -\bar{\varepsilon}_s \\ 2\bar{\varepsilon}_T \end{bmatrix} \mathrm{d}s - \begin{bmatrix} 2A_e \bar{\alpha} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \oint g_2 \, \mathrm{d}s. \tag{44}$$

Introduction of $\bar{\varkappa}_s$, $\bar{\varepsilon}_s$ and $\bar{\varepsilon}_T$ from (34b) and (35a, b) into (44) results in the relations

$$C_3 \begin{bmatrix} \bar{\alpha} \\ \bar{\varepsilon} \\ \bar{\varkappa}_x \\ \bar{\varkappa}_y \end{bmatrix} + C_4 \begin{bmatrix} \bar{S} \\ \overline{M} \\ \overline{N}_x \\ \overline{N}_y \end{bmatrix} = \oint [g_2 - Q_1^T (B_3 f_2 - B_4 f_1)] \, \mathrm{d}s. \tag{45}$$

Equations (43) and (45) determine the eight constants of integration in terms of integrals of starred quantities which are implicitly given by vectors f_i and g_i.

We note that the system of equations (43) and (45) for closed tubes may be reduced to a system of equations for open tubes upon appropriately interpreting the constitutive coefficients. Stipulating that an open-cross-section tube may be considered as if it were a closed-cross-section tube with an open part, we set in the constitutive equations (34a, b) the elements of B_1 equal to zero, and the elements of B_4 equal to infinite over the open part of the closed-cross-section. When this is done, the elements of C_3 and of the right-hand side of (45) are finite† and the elements of C_4 become infinite, requiring that the constants $\bar{S}$, $\overline{M}$, $\overline{N}_x$, $\overline{N}_y$ must vanish. Setting $\bar{S} = \overline{M} = \overline{N}_x = \overline{N}_y = 0$ into (43), we directly obtain a system of equations for the remaining four constants $\bar{\alpha}$, $\bar{\varepsilon}$, $\bar{\varkappa}_x$ and $\bar{\varkappa}_y$, of exactly the same form as equations (41) which follows from a direct consideration.

† In view of equation (45), the statement that the elements of the right side of (45) are finite involves the constitutive matrix B_4, the elements of which are stipulated to be infinite over the open part. However, we have with the relations $S^* = \int_{s_1}^{s_2} N_z^* \, \mathrm{d}s = \int_{s_1}^{s_2} 2M_T^* \, \mathrm{d}s = 0$, that the elements of f_1 in (37a) for open-cross-section tubes vanish over the open part. With this, the elements of $B_4 f_1$ may be stipulated to be finite over the open part of the closed-cross-section tube. The integral of $Q_1^T B_4 f_1$ on the right-hand side of (45) therefore has a finite value confirming our earlier observations.

EXPLICIT SOLUTION FOR A CLASS OF ANISOTROPIC SHELLS

In order to illustrate the nature of our general results, we consider shells with constitutive equations of the form

$$\begin{aligned} \varepsilon_z &= \frac{N_z}{C_{zz}}+\frac{N_s}{C_{zs}}+\frac{N_T}{C_{zT}}, & M_z &= D_{zz}\varkappa_z+D_{zs}\varkappa_s+D_{zT}2\varkappa_T, \\ \varepsilon_s &= \frac{N_z}{C_{sz}}+\frac{N_s}{C_{ss}}+\frac{N_T}{C_{sT}}, & M_s &= D_{sz}\varkappa_z+D_{ss}\varkappa_s+D_{sT}2\varkappa_T, \\ 2\varepsilon_T &= \frac{N_z}{C_{Tz}}+\frac{N_s}{C_{Ts}}+\frac{N_T}{C_{TT}}, & M_T &= D_{Tz}\varkappa_z+D_{Ts}\varkappa_s+D_{TT}2\varkappa_T, \end{aligned} \tag{46}$$

in conjunction with

$$\begin{aligned} P_s = P_z = 0, \qquad \gamma_z = \gamma_s = 0, \\ M_{sz} = M_{zs} \equiv M_T, \qquad \varepsilon_{zs} = \varepsilon_{sz} \equiv \varepsilon_T, \end{aligned} \tag{47}$$

as previously done for the problems of stretching, bending and twisting [3]. With this we consider separately solutions for the case of open-cross-section shells and for the case of closed-cross-section shells.

For both cases we can assume that the starred (z-dependent) portion of the state of stress and strain is given by corresponding results in [3], upon replacing (N, M_x, M_y, T) by (0, Q_y, Q_x, 0). In what follows we determine the barred (z-independent) portion of the stress and strain state, separately for open and for closed-cross-section shells. For the former case we list, in particular, the results for the special case of a flat plate.

Open-cross-section shells

Since $S^* = \bar{M} = \bar{N}_x = \bar{N}_y = \bar{S} = 0$, we have from equation (35b) that $\bar{M}_s$, $\bar{N}_s$ and $\bar{N}_{sz}$ are explicitly given by

$$\bar{M}_s = -\int_{s_1}^{s} 2M_T^* \, ds, \qquad \bar{N}_s = 0, \qquad \bar{N}_{sz} = -\int_{s_1}^{s} N_z^* \, ds. \tag{48}$$

We next use equations (46) together with the relations $\bar{N}_T = \bar{N}_{sz}+\frac{1}{2}k\bar{M}_T$ and $\bar{\varkappa}_T = \bar{\varkappa}_{zs}-\frac{1}{2}k\bar{\varepsilon}_T$ to express $\bar{N}_z$, $\bar{M}_z$ and $\bar{M}_T$ in terms of $\bar{\varepsilon}_z$, $\bar{\varkappa}_z$, $\bar{\varkappa}_{zs}$ and of the quantities given by (48).

After some transformations, the barred portion of equations (46) leads to relations which, with a simplification basic on the stipulations $D = O(Eh^3)$, $C = O(Eh)$, and within the range of applicability of shell theory $k^2h^2 \ll 1$, $|k\varepsilon_z| \ll |\varkappa_{zs}|$, and $\max(|\varkappa_{zs}|, |\varkappa_z|) \ll |k\varepsilon_z|/k^2h^2$, in accordance with [3], are of the form

$$\bar{M}_T = D_{TT}\left(1-\frac{D_{Ts}D_{sT}}{D_{TT}D_{ss}}\right)2\bar{\varkappa}_{zs}+D_{Tz}\left(1-\frac{D_{Ts}D_{sz}}{D_{Tz}D_{ss}}\right)\bar{\varkappa}_z-2\frac{D_{Ts}}{D_{ss}}\int_{s_1}^{s} M_T^* \, ds, \tag{49a}$$

$$\bar{M}_z = D_{zz}\left(1-\frac{D_{sz}D_{zs}}{D_{ss}D_{zz}}\right)\bar{\varkappa}_z+D_{zT}\left(1-\frac{D_{sT}D_{zs}}{D_{ss}D_{zT}}\right)2\bar{\varkappa}_{zs}-2\frac{D_{zs}}{D_{ss}}\int_{s_1}^{s} M_T^* \, ds, \tag{49b}$$

and

$$\bar{N}_z = C_{zz}\bar{\varepsilon}_z+\frac{C_{zz}}{C_{zT}}\int_{s_1}^{s} N_z^* \, ds. \tag{49c}$$

In this $\bar{\varkappa}_{zs}$, $\bar{\varkappa}_z$ and $\bar{\varepsilon}_z$ are given by (35a), rewritten here in the form

$$\begin{aligned} \bar{\varepsilon}_z &= \bar{\varepsilon} + \bar{\varkappa}_x y + \bar{\varkappa}_y x + \int_{s_1}^{s} (2\varepsilon_T^* - \alpha^* r_n)\, \mathrm{d}s, \\ \bar{\varkappa}_z &= \bar{\varkappa}_y y' - \bar{\varkappa}_x x' + \alpha^* r_t, \qquad \bar{\varkappa}_{zs} = \bar{\alpha} + \int_{s_1}^{s} \varkappa_s^*\, \mathrm{d}s. \end{aligned} \tag{50}$$

Equations (49a, b, c) together with (50) in turn are introduced into (10ā, b̄, c̄, d̄') to form a system of four simultaneous equations for $\bar{\varepsilon}$, $\bar{\alpha}$, $\bar{\varkappa}_x$ and $\bar{\varkappa}_y$. We list here, in particular, the system of equations for a *flat plate*, for which $s = x$, $x' = 1$, $y = y' = 0$, and $s_1 = -b$, $s_2 = b$,

$$\bar{\varepsilon}\int_{-b}^{b} C_{zz}\,\mathrm{d}x + \bar{\varkappa}_y \int_{-b}^{b} C_{zz}\, x\,\mathrm{d}x = -\int_{-b}^{b}\left(\frac{C_{zz}}{C_{zT}}\int_{-b}^{x} N_z^*\,\mathrm{d}x + C_{zz}\int_{-b}^{x} 2\varepsilon_T^*\,\mathrm{d}x\right)\mathrm{d}x, \tag{51a}$$

$$\bar{\varepsilon}\int_{-b}^{b} C_{zz}x\,\mathrm{d}x + \bar{\varkappa}_y \int_{-b}^{b} C_{zz}x^2\,\mathrm{d}x = -\int_{-b}^{b}\left(\frac{C_{zz}}{C_{zT}}\int_{-b}^{x} N_z^*\,\mathrm{d}x + C_{zz}\int_{-b}^{x} 2\varepsilon_T^*\,\mathrm{d}x\right)x\,\mathrm{d}x, \tag{51b}$$

$$\begin{aligned} &\bar{\varkappa}_x \int_{-b}^{b}\left(D_{zz} - \frac{D_{zs}D_{sz}}{D_{ss}}\right)\mathrm{d}x - 2\bar{\alpha}\int_{-b}^{b}\left(D_{zT} - \frac{D_{zs}D_{sT}}{D_{ss}}\right)\mathrm{d}x \\ &\quad = \int_{-b}^{b}\left[\left(D_{zz} - \frac{D_{zs}D_{sz}}{D_{ss}}\right)\alpha^* x + 2\left(D_{zT} - \frac{D_{zs}D_{sT}}{D_{ss}}\right)\int_{-b}^{x}\varkappa_s^*\,\mathrm{d}x - 2\frac{D_{zs}}{D_{ss}}\int_{-b}^{x} M_T^*\,\mathrm{d}x\right]\mathrm{d}x, \end{aligned} \tag{51c}$$

$$\begin{aligned} &4\bar{\alpha}\int_{-b}^{b}\left(D_{TT} - \frac{D_{Ts}D_{sT}}{D_{ss}}\right)\mathrm{d}x - 2\bar{\varkappa}_x\int_{-b}^{b}\left(D_{Tz} - \frac{D_{Ts}D_{sz}}{D_{ss}}\right)\mathrm{d}x \\ &\quad = \int_{-b}^{b}\left[xM_z^* - 4\left(D_{TT} - \frac{D_{sT}D_{Ts}}{D_{ss}}\right)\int_{-b}^{x}\varkappa_s^*\,\mathrm{d}x - 2\left(D_{Tz} - \frac{D_{Ts}D_{sz}}{D_{ss}}\right)\alpha^* x \right. \\ &\qquad \left. + 4\frac{D_{Ts}}{D_{ss}}\int_{-b}^{x} M_T^*\,\mathrm{d}x\right]\mathrm{d}x. \end{aligned} \tag{51d}$$

We note that the constants $\bar{\varepsilon}$ and $\bar{\varkappa}_y$ depend only on the stretching stress resultant N_z^* and the shearing strain resultant ε_T^*, while $\bar{\varkappa}_x$ and $\bar{\alpha}$ depend on the twisting and bending stress couples M_T^*, M_z^* and the twisting and bending strain couples α^*, $\varkappa_s^*$.

Having obtained $\bar{\varepsilon}$, $\bar{\varkappa}_x$, $\bar{\varkappa}_y$ and $\bar{\alpha}$, it becomes a simple matter to determine $\bar{\varepsilon}_z$, $\bar{\varkappa}_z$ and $\bar{\varkappa}_{zs}$ from (50) and $\overline{M}_T$, $\overline{M}_z$ and $\overline{N}_z$ from (49). The remaining quantities $\bar{\varepsilon}_T$, $\bar{\varepsilon}_s$ and $\bar{\varkappa}_s$ which incorporate the same simplification as (49a, b, c) are then obtained from

$$\bar{\varkappa}_s = -\frac{D_{sz}\bar{\varkappa}_z + D_{sT}2\bar{\varkappa}_{zs}}{D_{ss}} - \frac{1}{D_{ss}}\int_{-b}^{x} 2M_T^*\,\mathrm{d}x, \tag{52a}$$

$$\bar{\varepsilon}_s = \frac{C_{zz}}{C_{sz}}\bar{\varepsilon}_z - \frac{1}{C_{sT}}\left(1 - \frac{C_{sT}C_{zz}}{C_{sz}C_{zT}}\right)\int_{-b}^{x} N_z^*\,\mathrm{d}x. \tag{52b}$$

$$2\bar{\varepsilon}_T = \frac{C_{zz}}{C_{Tz}}\bar{\varepsilon}_z - \frac{1}{C_{TT}}\left(1 - \frac{C_{TT}C_{zz}}{C_{Tz}C_{zT}}\right)\int_{-b}^{x} N_z^*\,\mathrm{d}x. \tag{52c}$$

Closed-cross-section shells

In order to simplify the result for stresses and strains for closed-cross-section shells, we start with the same stipulations as in [3] that we may assume M_T, M_z and N_s negligible in the constitutive equations (46). With this, we have now as effective constitutive equations

$$M_z = 0, \qquad M_T = 0, \qquad \varkappa_s = \frac{1}{D}M_s, \tag{53a}$$

and

$$\begin{aligned} N_z &= C_{zz}\varepsilon_z - \frac{C_{zz}}{C_{zT}}N_{sz}, \\ \varepsilon_s &= \frac{C_{zz}}{C_{sz}}\varepsilon_z + \frac{1}{C_{sT}}\left(1 - \frac{C_{zz}C_{sT}}{C_{sz}C_{zT}}\right)N_{sz}, \\ 2\varepsilon_T &= \frac{C_{zz}}{C_{Tz}}\varepsilon_z + \frac{1}{C_{TT}}\left(1 - \frac{C_{zz}C_{TT}}{C_{Tz}C_{zT}}\right)N_{sz}. \end{aligned} \tag{53b}$$

We note that by letting $M_T = 0$, the possibility of treating the open-cross-section shell as a limiting case of the closed-cross-section shell is precluded.

In the following, we determine barred stresses and strains in terms of starred stresses and strains. We first have the barred portion of M_s, ε_z and N_{sz}, as in equations (35a, b) with $M_T^* = S^* = 0$,† written here as

$$\begin{aligned} \overline{M}_s &= \overline{M} + \overline{N}_x y + \overline{N}_y x, \qquad \overline{N}_{sz} = \bar{S} - \int_{s0}^{s} N_z^* \, ds. \\ \bar{\varepsilon}_z &= \bar{\varepsilon} + \bar{\varkappa}_x y + \bar{\varkappa}_y x + \int_{s0}^{s} (2\varepsilon_T^* - \alpha^* r_n) \, ds. \end{aligned} \tag{54}$$

We now introduce equations (53a, b) together with (54) into the overall equilibrium equations ($10\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}'$) and the strain integral relations ($9\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}'$) to form, as a special case of the system (43) and (45), a system of eight equations for the constants $\overline{M}$, $\overline{N}_x$, $\overline{N}_y$, $\bar{S}$, $\bar{\varepsilon}$, $\bar{\varkappa}_x$, $\bar{\varkappa}_y$ and $\bar{\alpha}$. This system of equations which with coordinate axes chosen so as to make $\oint C_{zz}(x, y, xy)\,ds = (0, 0, 0)$ appears in the form

$$\bar{\varepsilon}\oint C_{zz}\,ds - \bar{S}\oint (C_{zz}/C_{zT})\,ds = -\oint \left[\int_{s0}^{s} (2\varepsilon_T^* - \alpha^* r_n)\,ds + \left(\int_{s0}^{s} N_z^*\,ds\right)\Big/ C_{zT}\right] C_{zz}\,ds. \tag{55a}$$

$$\begin{aligned} &\bar{\varkappa}_x \oint C_{zz} y^2 \,ds - \bar{S}\oint (C_{zz}/C_{zT}) y\,ds \\ &\quad = -\oint \left[\int_{s0}^{s} (2\varepsilon_T^* - \alpha^* r_n)\,ds + \left(\int_{s0}^{s} N_z^*\,ds\right)\Big/ C_{zT}\right] C_{zz} y\,ds, \end{aligned} \tag{55b}$$

$$\begin{aligned} &\bar{\varkappa}_y \oint C_{zz} x^2 \,ds - \bar{S}\oint (C_{zz}/C_{zT}) x\,ds \\ &\quad = -\oint \left[\int_{s0}^{s} (2\varepsilon_T^* - \alpha^* r_n)\,ds + \left(\int_{s0}^{s} N_z^*\,ds\right)\Big/ C_{zT}\right] C_{zz} x\,ds, \end{aligned} \tag{55c}$$

$$2A_e\bar{S} = \oint \left(\int_{s0}^{s} N_z^*\,ds\right) r_n\,ds, \tag{55d}$$

† $S^*(\equiv N_{sz}^*) = 0$ is a direct consequence of equation (48d′) in [3], with T set equal to zero.

and

$$\overline{M}\oint\frac{\mathrm{d}s}{D}+\overline{N}_x\oint\frac{y\,\mathrm{d}s}{D}+\overline{N}_y\oint\frac{x\,\mathrm{d}s}{D}=0, \tag{56a}$$

$$\begin{aligned}&\overline{M}\oint\frac{y\,\mathrm{d}s}{D}+\overline{N}_x\oint\frac{y^2\,\mathrm{d}s}{D}+\overline{N}_y\oint\frac{xy\,\mathrm{d}s}{D}+\bar{S}\oint\left(1-\frac{C_{zz}C_{sT}}{C_{sz}C_{zT}}\right)\frac{x'\,\mathrm{d}s}{C_{sT}}\\&\quad+\bar{\varepsilon}\oint(C_{zz}/C_{sz})x'\,\mathrm{d}s+\bar{\varkappa}_x\oint(C_{zz}/C_{sz})x'y\,\mathrm{d}s+\bar{\varkappa}_y\oint(C_{zz}/C_{sz})xx'\,\mathrm{d}s\\&\quad=\oint\left\{\frac{1}{C_{sT}}\left(1-\frac{C_{zz}C_{sT}}{C_{sz}C_{zT}}\right)\int_{s_0}^{s}N_z^*\,\mathrm{d}s-\frac{C_{zz}}{C_{sz}}\int_{s_0}^{s}(2\varepsilon_T^*-\alpha^*r_n)\,\mathrm{d}s\right\}x'\,\mathrm{d}s,\end{aligned} \tag{56b}$$

$$\begin{aligned}&\overline{M}\oint\frac{x\,\mathrm{d}s}{D}+\overline{N}_x\oint\frac{xy\,\mathrm{d}s}{D}+\overline{N}_y\oint\frac{x^2\,\mathrm{d}s}{D}-\bar{S}\oint\left(1-\frac{C_{zz}C_{sT}}{C_{sz}C_{zT}}\right)\frac{y'\,\mathrm{d}s}{C_{sT}}\\&\quad-\bar{\varepsilon}\oint(C_{zz}/C_{sz})y'\,\mathrm{d}s-\bar{\varkappa}_x\oint(C_{zz}/C_{sz})y'y\,\mathrm{d}s-\bar{\varkappa}_y\oint(C_{zz}/C_{sz})xy'\,\mathrm{d}s\\&\quad=-\oint\left\{\frac{1}{C_{sT}}\left(1-\frac{C_{zz}C_{sT}}{C_{sz}C_{zT}}\right)\int_{s_0}^{s}N_z^*\,\mathrm{d}s-\frac{C_{zz}}{C_{sz}}\int_{s_0}^{s}(2\varepsilon_T^*-\alpha^*r_n)\,\mathrm{d}s\right\}y'\,\mathrm{d}s,\end{aligned} \tag{56c}$$

$$\begin{aligned}&-2A_e\bar{\alpha}+\bar{\varepsilon}\oint(C_{zz}/C_{Tz})\,\mathrm{d}s+\bar{\varkappa}_x\oint(C_{zz}/C_{Tz})y\,\mathrm{d}s+\bar{\varkappa}_y\oint(C_{zz}/C_{Tz})x\,\mathrm{d}s+\bar{S}\oint\left(1-\frac{C_{zz}C_{TT}}{C_{Tz}C_{zT}}\right)\frac{\mathrm{d}s}{C_{TT}}\\&\quad=\oint\left\{r_n\int_{s_0}^{s}\varkappa_s^*\,\mathrm{d}s-r_t\varepsilon_s^*+\frac{1}{C_{TT}}\left(1-\frac{C_{zz}C_{TT}}{C_{Tz}C_{zT}}\right)\int_{s_0}^{s}N_z^*\,\mathrm{d}s-\frac{C_{zz}}{C_{Tz}}\int_{s_0}^{s}(2\varepsilon_T^*-\alpha^*r_n)\,\mathrm{d}s\right\}\mathrm{d}s.\end{aligned} \tag{56d}$$

It is apparent that equation (55d) gives $\bar{S}$ explicitly in term of N_z^*. Introducing this result into (55a, b, c) we obtain $\bar{\varepsilon}$, $\bar{\varkappa}_x$ and $\bar{\varkappa}_y$. We then use equation (56d) to express $\bar{\alpha}$ in terms of $\varkappa_s^*$, ε_s^*, ε_T^*, α^* and N_z^*. With the determined results for $\bar{S}$, $\bar{\varepsilon}$, $\bar{\varkappa}_x$ and $\bar{\varkappa}_y$, equations (56a, b, c) then become three equations for $\overline{M}$, $\overline{N}_x$ and $\overline{N}_y$.

Having determined the eight constants, it becomes a simple matter to obtain the barred measures of stress and strain upon making use of the first integrals, (54) together with $\bar{\varkappa}_z$, $\bar{\varkappa}_{zs}$ and $\overline{N}_s$ in (35a, b), and of the constitutive equations (53a, b). Among these results, we are particularly interested in the distribution of the shearing stress resultant $\overline{N}_{sz}(=N_{sz}=N_{zs}$ because $N_{sz}^*=M_T=0)$ since an explicit result for this has, so far as the author knows, not yet been given.

We first introduce $\bar{S}$ from (55d) into (54) to obtain

$$N_{zs}=\frac{1}{2A_e}\oint\left(\int_{s_0}^{s}N_z^*\,\mathrm{d}s\right)r_n\,\mathrm{d}s-\int_{s_0}^{s}N_z^*\,\mathrm{d}s. \tag{57}$$

Here the initial arc length s_0 in (57) is arbitrary because N_{zs} is independent of it.

We now rewrite equation (57) in more explicit form, upon making use of $N_z^*=C_{zz}\varepsilon_z^*=C_{zz}(\varepsilon^*+\varkappa_x^*y+\varkappa_y^*x)$ and of $\varepsilon^*=0$, $\varkappa_x^*\oint C_{zz}y^2\,\mathrm{d}s=Q_y$ and $\varkappa_y^*\oint C_{zz}x^2\,\mathrm{d}s=Q_x$.† By doing this, the result, with a suitable integration by part and with Q_x set equal to zero for

† The values of ε^*, $\varkappa_x^*$ and $\varkappa_y^*$ are obtained from equations (48a′, b′, c′) in [3] with N, M_x and M_y replaced by 0, Q_y and Q_x.

simplicity's sake, appears in the form

$$N_{zs} = -\frac{Q_y}{\oint C_{zz}y^2\,ds}\left\{\oint\left(\frac{\int r_n\,ds}{\oint r_n\,ds}\right)C_{zz}y\,ds+\int C_{zz}y\,ds\right\}. \tag{58}$$

It is of interest to note that N_{zs} in equation (58) depends only on the constitutive coefficient C_{zz} even though a system of anisotropic constitutive equations (53a, b) has been used.

Uniform closed-cross-section circular cylindrical shell

We set $x = a\sin\varphi$, $y = -a\cos\varphi$ and $ds = a\,d\varphi$, and replace (N, M_x, M_y, T) in [3] by $(0, Q_y, 0, 0)$. In this way we obtain directly from equations (48a′, b′, c′, d′) and (51a, b, c, d) in [3],

$$\varepsilon^* = \varkappa_y^* = \alpha^* = 0, \qquad \varkappa_x^* = \frac{Q_y}{\pi C_{zz}a^3}, \tag{59a}$$

$$M^* = N_y^* = S^* = 0, \qquad N_x^* = \frac{Q_y}{\pi a^2}\frac{D}{C_{sz}a^2}. \tag{59b}$$

Equations (59a, b) together with equations (47c′) and (49a′, b′, c′) in [3] give for stress quantities

$$N_{sz}^* = 0, \qquad (N_z^*, N_s^*, M_s^*) = \frac{Q_y}{\pi a^3}\left(1, -\frac{D}{C_{sz}a^2}, \frac{D}{C_{sz}a}\right)y, \tag{60}$$

and for strain quantities

$$\begin{aligned}\varkappa_{zs}^* = 0, \qquad (\varkappa_z^*, \varkappa_s^*) &= \frac{Q_y}{\pi a^4}\left(\frac{1}{C_{zz}}, \frac{1}{C_{sz}}\right)y,\\ (\varepsilon_z^*, \varepsilon_s^*, 2\varepsilon_T^*) &= \frac{Q_y}{\pi a^3}\left(\frac{1}{C_{zz}}, \frac{1}{C_{sz}}, \frac{1}{C_{Tz}}\right)y.\end{aligned} \tag{61}$$

We now make use of equations (53) to (56) to write the z-independent measures of stress and strain as follows. The constants $\bar{\varepsilon}$, $\bar{\varkappa}_x$, $\bar{\varkappa}_y$, $\bar{\alpha}$, $\overline{M}$, $\overline{N}_x$, $\overline{N}_y$ and $\bar{S}$ are, with s_0 set equal to zero, given by

$$\begin{aligned}\bar{\varepsilon} = \bar{\varkappa}_x = \bar{\alpha} = 0, \qquad \bar{\varkappa}_y &= \frac{Q_y}{\pi a^2}\left(\frac{1}{C_{Tz}}+\frac{1}{C_{zT}}\right),\\ \overline{M} = \overline{N}_x = \bar{S} = 0, \qquad \overline{N}_y &= \frac{Q_y}{\pi a}\frac{D}{C_{sT}a^2}.\end{aligned} \tag{62}$$

Stress resultants and couples are

$$\overline{N}_z = 0, \qquad \overline{N}_{sz} = \frac{Q_y x}{\pi a^2}, \qquad \overline{N}_s = -\frac{\overline{M}_s}{a} = -\frac{Q_y x}{\pi a^2}\frac{D}{C_{sT}a^2}. \tag{63}$$

Strain resultants and couples are

$$\begin{aligned}(\bar{\varepsilon}_z, \bar{\varepsilon}_s, 2\bar{\varepsilon}_T) &= \frac{Q_y}{\pi a^2}\left(\frac{1}{C_{zT}}, \frac{1}{C_{sT}}, \frac{1}{C_{TT}}\right)x,\\ (\bar{\varkappa}_z, \bar{\varkappa}_s, \bar{\varkappa}_{zs}) &= \frac{Q_y}{\pi a^3}\left(\frac{1}{C_{Tz}}+\frac{1}{C_{zT}}, \frac{1}{C_{sT}}, -\frac{1}{C_{sz}}\right)x.\end{aligned} \tag{64}$$

With (60), (61), (63) and (64), we have as expressions for measures of stress and strain

$$N_z = \frac{Q_y}{\pi a^3} yz, \qquad N_{sz} = \frac{Q_y}{\pi a^2} x,$$
$$N_s = -\frac{M_s}{a} = -\frac{Q_y}{\pi a^3}\frac{D}{C_{sz}a^2}\left(yz + ax\frac{C_{sz}}{C_{sT}}\right), \tag{65}$$

and

$$\varepsilon_z = \frac{Q_y}{\pi a^3}\left(\frac{yz}{C_{zz}} + \frac{ax}{C_{zT}}\right), \qquad \varepsilon_s = \frac{Q_y}{\pi a^3}\left(\frac{yz}{C_{sz}} + \frac{ax}{C_{sT}}\right),$$
$$2\varepsilon_T = \frac{Q_y}{\pi a^3}\left(\frac{yz}{C_{Tz}} + \frac{ax}{C_{TT}}\right), \qquad \varkappa_z = \frac{Q_y}{\pi a^4}\left(\frac{yz}{C_{zz}} + \frac{ax}{C_{Tz}} + \frac{ax}{C_{zT}}\right), \tag{66}$$
$$\varkappa_s = \frac{Q_y}{\pi a^4}\left(\frac{yz}{C_{sz}} + \frac{ax}{C_{sT}}\right), \qquad \varkappa_{zs} = -\frac{Q_y}{\pi a^3}\frac{x}{C_{sz}}.$$

The results for N_z and N_{sz} are of exactly the same form as those following from elementary considerations. The quantities N_s and M_s are of negligible magnitude, the same as for the problem considered in [3].

The quantities ε_z, ε_s, $\varkappa_z$ and $\varkappa_s$ consist of two portions, the portion linear on z and the portion independent of z. The latter portion is due to the effect of anisotropy. When $z/a \gg 1$, the effect of anisotropy on ε_z, ε_s, $\varkappa_z$ and $\varkappa_s$ becomes unimportant. The strain resultant ε_T also consists of two portions. But in ε_T the anisotropic effect which occurs in the portion depending on z dominates. The strain couple $\varkappa_{zs}$ comes out independent of anisotropic constitutive coefficients.

The results which are probably of greatest interest for the present case are expressions for the displacement components u_z, v_x and v_y. With x_0 and y_0 set equal to zero and with u_z, v_x and v_y assumed to vanish at $y = z = 0$, we obtain from equations (22a) and (24a, b),

$$u_z = \frac{z^2}{2}\frac{Q_y y}{\pi a^3 C_{zz}} + z\frac{Q_y x}{\pi a^2 C_{zT}} + \frac{Q_y}{\pi a}\left(\frac{1}{C_{TT}} + \frac{1}{C_{sz}}\right)y, \tag{67a}$$

$$v_x = -\frac{z^2}{2}\frac{Q_y}{\pi a^2}\left(\frac{1}{C_{Tz}} + \frac{1}{C_{zT}}\right) + z\frac{Q_y xy}{\pi a^3 C_{sz}} - \frac{Q_y y^2}{\pi a^2 C_{sT}}, \tag{67b}$$

$$v_y = -\frac{z^3}{6}\frac{Q_y}{\pi a^3 C_{zz}} - z\frac{Q_y x^2}{\pi a^3 C_{sz}} + \frac{Q_y xy}{\pi a^2 C_{sT}}. \tag{67c}$$

Equation (67a) indicates that the effect of anisotropy on u_z is not so important as the first term when z/a is large. Equation (67b) shows that the leading term of v_x depends on z^2 and is due to anisotropy. Comparing this term with the second term we find that the lateral contraction due to the constitutive coefficient C_{sz} becomes of secondary importance when $z/a \gg 1$. Equation (67c) indicates that the effect of anisotropy on v_y represents a distortion which is unimportant insofar as the displacements for the entire shell are concerned.

In order to see the effects of C_{TT} and C_{sz} on v_y, we first observe equation (67a). Since the last term in (67a) represents a rigid body displacement, we may rotate the entire beam about the x-axis so as to make $u_z = 0$ at the plane $z = 0$. When this has been done, $\partial u_z/\partial y$

at $z = 0$ becomes zero and consequently v_y is increased by a quantity $Q_y z(1/C_{TT}+1/C_{sz})/\pi a$. We then obtain v_y, with the distortion term in (67c) discarded and with $a^2-x^2 = y^2$, as

$$v_y = -\frac{z^3}{6}\frac{Q_y}{\pi a^3 C_{zz}}\left(1-6\frac{a^2}{z^2}\frac{C_{zz}}{C_{TT}}-6\frac{y^2}{z^2}\frac{C_{zz}}{C_{sz}}\right). \tag{68}$$

In the parenthesis of equation (68), the second term represents the well-known effect of shearing force. The last term represents the effect of lateral contraction.

The displacement component v_x, with the distortion term discarded, is listed in the form

$$v_x = -\frac{z^2}{2}\frac{Q_y}{\pi a^2}\left(\frac{1}{C_{Tz}}+\frac{1}{C_{zT}}\right)\left[1-\frac{xy}{za}\frac{C_{Tz}C_{zT}}{C_{sz}(C_{Tz}+C_{zT})}\right]. \tag{69}$$

Remarkably, the effect of anisotropy on v_x is large in comparison with the effect of shearing force on v_y, yet is small in comparison with the first term in (68) when z/a is large.

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Абстракт—Исследуется задача определения напряжений и деформаций в анизотрнпных, упругих. осебо однородных, тонкостенных оболочках, подверженных действиы равных и противоположно направленных поперечных краевых усилий и добавочных моментов, для обеспечения полного равновесия. Метод решения основан на разложении напряжений и деформаций в члены частично зависимые линейно от осевой координаты и частично независимые от осевой координаты. Наиболее значительным аспектом работы является обуждение эффекта анизотропии материала. Даются иллюстрации общих формул теории для класса оболочек, изготовленных из "обыкновенного" материала. Для этого случая, получаются формулы в явной форме, которые касаются некоторых типов оболочек, как открытого так и замкнутого поперечного сечения.